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Localization and solitary waves in solid mechanics

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1. Introduction

A response of a physical system that is localized to some portion of a distinguished coordinate (either space or time) is surely the next most fundamental state after homogeneous equilibrium and periodicity. This collection of original research articles is dedicated to the description and analysis of spatially localized waves in solid mechanics. While acknowledging that sharply discontinuous localizations are also of physical importance, we concentrate here on those that arise as a balance between two conflicting influences, one pinching and the other dispersive, such that they are spread over a finite spatial domain—akin to plastic necking, for example, rather than brittle fracture. Applications range from the folding of geological strata and the buckling of cylindrical shells, twisted rods and pipelines, to the propagation of travelling solitary waves in suspended beam systems. All may be described by a new breed of mathematical theories based on the analysis of homoclinic solutions of differential equations posed on an infinite length scale. In truth no material length is infinite, but it sometimes makes sense to assume that a system sits within an infinite domain; this recognizes that if it is 'long enough', the significance of the boundary conditions is often swamped by the homoclinic influence.

The key to this form of localization is that the processes in question should be nonlinear. In elastic buckling studies for example (see the contribution of Lord *et al.* on cylindrical shells), classical linear theory describes buckling via a critical (spatially periodic) eigenmode associated with a zero eigenvalue. However the buckling of shells is markedly sub-critical so that buckling usually occurs via a violent jump to a finite amplitude state. Nonlinearities are then crucial in determining the shape of the post-buckled state, and a spatially localized form is more likely to be adopted because it requires less energy than a fully periodic one (see Hunt *et al.* (1989) for a more detailed argument along these lines). There is indeed a great deal of experimental evidence for localized buckling; see the references in the contributions by Champneys *et al.* and Lord *et al.* A key point is that such localization does not necessarily arise from localized imperfections, but exists as a solution of the underlying (perfect) differential equation. Imperfections do however have a role in breaking the translational invariance associated with homoclinic behaviour, and hence in deciding where the localization will be centred.

A solitary wave is another form of localized phenomenon when viewed in a coordinate system that travels at the wave speed. A classical example is a solution of

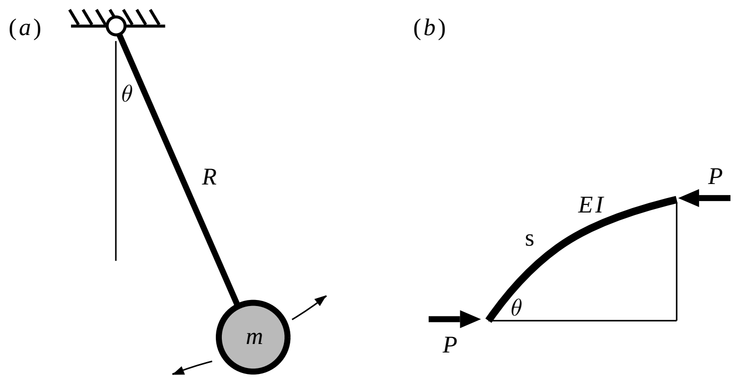

Figure 1. (*a*) Simple pendulum. (*b*) Strut element.

the Korteweg de Vries (KdV) equation (Korteweg & de Vries 1895) which was first derived to describe the observation by Scott Russell in 1834 of a solitary water wave travelling along a canal. Specifically, the KdV equation may be written as

$$u_t + 6uu_x + u_{xxx} = 0. \tag{1.1}$$

Moving to a travelling coordinate frame by writing $\xi = x + ct$, and taking a first integral leads to the ordinary differential equation

$$u_{\xi\xi} + cu + 3u^2 = 0, \tag{1.2}$$

which has a solitary wave solution $u = -(\frac{1}{2}c)\mathrm{sech}^2(\frac{1}{2}\sqrt{-c}\xi)$ for $c < 0$. Loosely speaking, solitary waves are born in a balance between dispersive effects (the u_{xxx} term in (1.1), acting to spread the response in x) and those that act to (thoroughly) localize or pinch (the nonlinear amplitude-dependent combination of u_t and uu_x terms). Solitary wave solutions of the KdV equation and the related nonlinear Schrödinger (NLS) equation have been of much interest in recent years, attracting the name 'soliton' for their remarkable particle-like properties. Solitons are found to be very stable and able to pass through one another with only a shift in phase. There is now a rich theory of solitons based on the completely integrable structure of the KdV, NLS and related equations (see, for example, Ablowitz & Clarkson 1991). Solitons also arise in applications including travelling waves of rods and beams (see, for example, Slepyan *et al.* (1995) and the reference list of Kehrbaum & Maddocks). However, such integrable systems are not of much concern in this issue. One of our aims is to show that solitary waves of non-integrable systems lead to even more interesting mathematics, with many open questions. Indeed, the contribution by Chen & McKenna suggests that solitary waves of a certain fourth-order hyperbolic equation share some of the remarkable interaction properties of solitons, without the system appearing to be completely integrable.

It is not altogether surprising that equation (1.2) appears in a coordinate system that combines space and time, since identical equations can be found in each independently. Consider for example the (dimensionless) equation for the frictionless simple pendulum of figure 1*a*,

$$\theta_{tt} = -\sin(\theta), \tag{1.3}$$

where $\theta(t)$ is the angle of swing. Solving this with initial conditions corresponding to the pendulum pointing downwards with exactly the right velocity to reach the top, $\{\theta(0), \dot{\theta}(0)\} = \{0, 2\}$, leads to a homoclinic solution with $\theta(t) \to \pi$ as $t \to \pm\infty$. Similarly, the spatial equilibrium configuration of the planar elastica of figure 1*b*, corresponding to an axially compressed elastic strut or column, can be described in terms of the angular deformation θ of the arc-length s along its centreline. After non-dimensionalization, a force balance leads to the equilibrium equation

$$\theta_{ss} = -\sin(\theta). \tag{1.4}$$

Hence spatial equilibrium solutions of an infinitely long elastic strut (the elastica) are in one-to-one correspondence with motions of a simple pendulum, with the homoclinic orbit of the pendulum corresponding to a localized loop of the elastica. This analogy between spatial and temporal problems dates back to Kirchhoff (1859); see Thompson & Virgin (1988) and the contribution of Domokos for more details. It occurs also in the simplest model of torsional rod buckling, as discussed in detail by Kehrbaum & Maddocks, where the equivalent dynamical problem is that of a spinning top.

If the $\sin(\theta)$ in (1.3) is replaced by the first two terms in the Taylor expansion of a more general nonlinear restoring force, then the quadratic Duffing equation is obtained:

$$\theta_{tt} + \theta + \tfrac{1}{2}\theta^2 = 0. \tag{1.5}$$

This has been used (with the addition of damping and forcing terms) to describe a range of nonlinear oscillations, for example ship capsize (Thompson 1996, 1997). In these studies, the bifurcations caused by the homoclinic orbit of (1.5) are shown to play an organizing role for chaotic dynamics and for basins of attraction of competing stable motions. Observe however, that apart from a scaling factor for the nonlinear term, (1.5) is identical to the travelling wave KdV equation (1.2). The variables have totally different meanings, but the homoclinic solution is important for both models, albeit for different reasons.

Another essential ingredient of the localization described in this issue, is that it typically occurs in systems for which there is some form of energy conservation. Such problems can often be posed in a variational formulation, leading via Euler–Lagrange equations to differential equations which are *Hamiltonian* (the analogue of a classical mechanical system described in terms of generalized coordinates and momenta). Reversibility (Devaney 1976*b*) is also a common feature of many of the differential equations. A recurring theme will be homoclinic orbits of reversible Hamiltonian systems of two degree of freedom. An archetype of such a system is the fourth-order equation

$$u_{xxxx} + Pu_{xx} + u = f(u), \tag{1.6}$$

where P is a parameter and $f(u)$ is a nonlinear function. The contribution by Sandstede considers this as a model for a compressed strut resting on a nonlinear elastic foundation. Chen & McKenna and Hunt & Blackmore take $f(u)$ to be piecewise linear, respectively, as a model for solitary waves of a suspended beam with zero stiffness once the suspension device goes slack and as a model for the equilibrium configuration of a pipeline which is free to lift off from a compressible foundation.

Equations such as (1.6) are known to admit infinitely many *multi-modal* homoclinic orbits, effectively comprising copies of a primary sech-like solution glued together at different separations (Champneys & Toland 1993). The key to the infinite multiplicity is that (for $-2 < P < 2$) the linearization at the trivial equilibrium of (1.6) has four complex eigenvalues; in this context it was proved by Devaney (1976*a*) that the existence of one non-degenerate homoclinic orbit implies chaos. For the kind of applications that we have in mind, this implies that multitudinous equilibrium solutions exist, all of which can satisfy imposed boundary conditions (see the contribution of Champneys *et al.* and references therein in the context of 'spatially chaotic' equilibrium configurations of twisted rods).

One of the practical messages of the issue is thus directed at numerical computation. It is argued that if localized solutions are required, numerical methods specifically designed for such purposes should be used. Straightforward 'black box' finite element analysis for example is likely to miss the solutions of practical significance, in the chaotic mêlée (see Hunt *et al.* (1997) for a more detailed discussion). The key to computing localized solutions with minimum error over a truncated interval is to prescribe boundary conditions that force decay at $x = \pm\infty$ with the right asymptotics (Beyn 1990; Friedman & Doedel 1991). Such an approach is adopted by many of the contributors below. Another interesting aspect of the numerical computation of localization is the possibility of the destruction or spurious creation of extra localized solutions under numerical discretization. The contribution of Domokos discusses this topic applied to the classical Euler strut problem (see also Fiedler & Scheurle (1996) for some interesting theoretical results).

As well as questions of existence and computation, those of stability are at least partially addressed. While one of the aims of this issue is to show that there is now some general theory for the first two of these, stability is apparently more context dependent. Sandstede for example considers the localized bucking pulses of (1.6) as an elastic strut model; stability is then determined by the second variation of the energy functional. For the travelling wave problem of Chen & McKenna it is governed by the full time-dependent partial differential equations (PDEs). In that of Hunt *et al.*, on geological strata, stability must be taken in a pseudo (secular) sense appropriate for slow evolution over time. Sandstede also discusses quite general conditions under which multi-modal homoclinic solutions, arising in a number of the models in this issue, may be stable.

For the sake of brevity, we have concentrated in this issue on localized phenomena in solid mechanics. However, localization of this kind appears important in many other disciplines as well. To mention just a few examples aside from the present applications, equation (1.6) with various nonlinear terms $f(u)$ arises in the description of solitary light pulses (see, for example, Buryak & Akhmediev 1995), of solitary water waves with surface tension (Buffoni *et al.* 1996) and in the governing of pattern formation (Peletier & Troy 1996). See Champneys (1997) for a more thorough review.

An issue of this nature cannot give a general picture. It is not a text book—rather it is a series of snapshots. Nevertheless we hope these give some guide to current and future directions of research in the topic of localization, of use to theoretician and practitioner alike.

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